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Effect of layering on ballistic properties of metallic shields against sharp-nosed rigid projectiles

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ABSTRACT

On the basis of general assumptions that are pertinent to penetration modeling, we investigated the effect of layering of metallic shields on their protective characteristics against sharp-nosed rigid impactors and established/explained a number of general features of the penetration phenomenon. The main results can be formulated as follows: (i) any layering has an adverse effect on protective properties of the shield against perforation, (ii) increasing the number of layers with the same thicknesses impairs protective properties of the shield. These assertions are validated by analyzing results of experiments and numerical calculations available in the literature that allow comparing protective properties of monolithic and layered shields.

1. Introduction

Effect of layering, i.e., replacing the monolithic plate by several plates in contact with the same total thickness, on the ballistic properties of metallic shields against rigid projectiles has attracted considerable interest of the researchers in the field for a long time. More detailed information can be found in the surveys [1,2], monograph [3] and recent publications [4–12].

Quite a few publications examine sharp-nosed impactors [4–10,13–18] that are considered in this study. Most of these studies report impairing of protective properties of the shield as a result of layering which is observed in experiments and numerical calculations. Using a semi-empirical model Radin and Goldsmith [14] proved that the ballistic limit velocity (BLV) of a two-layer shield with layers in contact and having the same thicknesses is smaller than the BLV of a monolithic shield manufactured from the same material and having the same total thickness. To the best of our knowledge the latter study is the only one which employed analytical approach.

In present study we investigate the effect of layering, namely, the thicknesses and the number of the layers, on the protective property of a shield. This study is based on the approach that we proposed and used recently to concrete shields [19]. The special feature of this approach is that it does not require a closed mathematical model for calculating BLV of monolithic and layered shields. We showed that many characteristics of layered shields can be explained and predicted on the basis of general properties of the functions which determine penetration models without invoking particular expressions for these functions. This is very important since qualitative behavior of the characteristics in penetration mechanics is generally determined using simple models which include approximations of the experimental data.

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Nomenclature

D	diameter of the cylindrical part of the impactor
g	function defined in Eq. (13)
G	function in Eq. (2)
$H^{(\nu\mu)},\ H^{(\nu)}$	elements of the Hessian
$k_1,\ k_2$	parameters of models
m	mass of the impactor
N	number of plates in the layered shield
T	thickness of a shield
$T_{\min}$	lower bound of the domain of function $G(T)$
T_{sum}	total thickness of the layers in the shield
$T^{(j)}$	thickness of the jth plate
v_{bl}	ballistic limit velocity of the monolithic shield
v_{res}	residual velocity for the monolithic shield
V_{bl}	ballistic limit velocity of the layered shield
$V_{bl}^{(j)}$	BLV of the jth plate
V_{imp}	impact velocity of the layered shield
$V_{imp}^{(j)}$	impact velocity of the jth plate
V_{res}	residual velocity of the layered shield
$V_{res}^{(j)}$	residual velocity of the jth plate
$\alpha,\ \beta$	parameters of the model in Eq. (16)
$\eta_{res},\ \eta_{bl}$	parameters in Eq. (17)
δ	parameter defined in Eq. (7)
Θ	nose angle for the conical-nosed projectile
Λ	criterion of optimization in Eq. (B.1)
$\mu_0,\ \mu_1$	parameter of the model in Eq. (10)
BLV	ballistic limit velocity

2. Mathematical models of the layered shield

Consider a high speed normal penetration of a rigid sharp-nosed striker into a shield consisting of N metal plates in contact. The plates are numbered in the direction of the impactor's motion as $1, 2, \ldots, N$ and the thickness of the ith plate is $T^{(i)}$. Based on the frequently used assumption that the layers do not interact and are perforated independently, the residual velocity at the jth plate $V_{res}^{(j)}$ is equal to the impact velocity at the $(j+1)$th plate $V_{imp}^{(j+1)}$ for $j = 1 \div N-1$. In addition, the impact velocity of the shield V_{imp} is equal to $V_{imp}^{(1)}$, and the residual velocity of the shield V_{res} is equal to $V_{res}^{(N)} = V_{imp}^{(N+1)}$ (the notation $V_{imp}^{(N+1)}$ is introduced here for convenience). Hereafter, the superscript in the brackets denotes the ordinal number of the layer.

We use the following relationship between the impact velocity $V_{imp}^{(j)}$, the residual velocity $V_{res}^{(j)}$ and the BLV of the plates [20]:

$$V_{imp}^{(j)2} - V_{res}^{(j)2} = V_{bl}^{(j)2}, \tag{1}$$

where the BLV is defined as the minimum impact velocity of the impactor required for perforating a shield whereby impactor emerges from the shield with a zero residual velocity.

Since the plates are manufactured from the same material, the BLV of a plate is a function of its thickness:

$$V_{bl}^{(j)2} = G(T^{(j)}), \quad G(0) = 0, \quad j = 1, 2, \ldots, N, \tag{2}$$

where G is an increasing function.

Summing Eq. (1) from $j = 1$ to $j = N$ yields:

$$V_{imp}^2 - V_{res}^2 = \sum_{j=1}^{N} G(T^{(j)}), \tag{3}$$

and, consequently, the expression for the BLV of the layered shield V_{bl} reads:

$$V_{bl}^2 = \sum_{j=1}^{N} G(T^{(j)}). \tag{4}$$

In the case of the monolithic shield with the thickness

$$T_{sum} = T^{(1)} + T^{(2)} + \cdots + T^{(N)}, \tag{5}$$

the relationship between the BLV, v_{bl}, and the thickness of the shield can be obtained from Eq. (4) substituting $V_{bl} = v_{bl},\ N = 1$ and $T^{(1)} = T_{sum}$:

$$v_{bl}^2 = G(T_{sum}) = G\left(\sum_{j=1}^{N} T^{(j)}\right). \tag{6}$$

3. Comparison of monolithic and layered shields

In this section, we compare the magnitudes of the BLVs of the layered shield and of the monolithic shield having the same thickness. Clearly, one can consider the difference:

$$\delta = v_{bl}^2 - V_{bl}^2 = G\left(\sum_{j=1}^{N} T^{(j)}\right) - \sum_{j=1}^{N} G(T^{(j)}), \tag{7}$$

rather than the difference of the BLVs, $v_{bl} - V_{bl}$. The main goal of this Section is to prove the following claim.

Claim. Let $G(T)$ be a convex (convex downwards), twice differentiable function which is defined in the interval $T \geqslant T_{\min} \geqslant 0$ and satisfies the inequality:

$$T_{\min} G'(T_{\min}) - G(T_{\min}) \geqslant 0, \tag{8}$$

where $T_{\min}$ is given. Then $\delta > 0$ for

$$T^{(j)} \geqslant T_{\min}, \quad T^{(j)} > 0, \quad j = 1, 2, \ldots, N. \tag{9}$$

The proof of above claim is given in Appendix A.

Introducing the parameter $T_{\min}$ allows us to separate the range of the considered plate thicknesses. If $T_{\min} = 0$, then $G(0) = 0$ as can be deduced by simple physically reasoning. When $T_{\min} > 0$ function $G(T)$ is not defined for $T < T_{\min}$ and formal extension of this dependence for $T < T_{\min}$ does not have physical sense and often implies that $G(0) \neq 0$. Geometrically, Eq. (8) can be interpreted as follows. The tangent at the point $T = T_{\min}$ to the curve determined by the function $w = G(T)$ in the T, w plane intersects the axis $T = 0$ at the point $w_0 \leqslant 0$. In particular, this condition reduces to the inequality $G(0) \leqslant 0$ if $T_{\min} = 0$. Eq. (8) can be changed by continuing the function $w = G(T)$ up to $T = 0$ and retaining the convexity so that it intersects the axis $T = 0$ at $w_0 \leqslant 0$.

If $G(T)$ is a linear function,

$$G(T) = \mu_1 T + \mu_0, \tag{10}$$

then

$$\delta = \left(\mu_1 \sum_{j=1}^{N} T^{(j)} + \mu_0\right) - \sum_{j=1}^{N} (\mu_1 T^{(j)} + \mu_0) = \mu_0 (1 - N). \tag{11}$$

Thus $\delta > 0$ if $\mu_0 < 0$ and $\delta = 0$ if $\mu_0 = 0$.

Consequently, the monolithic shield is superior over any layered shield with the same total thickness if $G(T)$ is convex function and conditions given by Eqs. (8) and (9) are satisfied or $G(T)$ is a linear function with negative constant term, μ_0. If $G(T) \propto T$, then layering does not effect ballistic properties of the shield.

4. The worst layering for a given number of layers

According to the above analysis, ballistic properties of the shield can be impaired by layering. Let us estimate the maximum reduction of the ballistic performance of the shield. To this end, we will show that this occurs when the layers have the same thicknesses. We assume here that the number of the plates in the shield N and the total thickness of the shield T_{sum} are given. The problem is to minimize V_{bl} in Eq. (4) under constraints given by Eqs. (5) and (9). We also assume that $T_{sum}/N \geqslant T_{\min}$, i.e., the thicknesses $T^{(j)} = T_{sum}/N,\ j = 1, 2, \ldots, N$ are permissible.

Investigation of this optimization problem with convex function $G(T)$, implies (see Appendix B) that the worst layering occurs when the monolithic shield is replaced by the identical plates having the same total thickness. If $G(T)$ is a linear function given by Eq. (10) then

$$V_{bl}^2 = \sum_{j=1}^{N} (\mu_1 T^{(j)} + \mu_0) = \mu_1 T_{sum} + \mu_0 N, \tag{12}$$

i.e., sizes of the layers do not affect ballistic properties of the shield.

5. Effect of the number of layers

Let us determine the dependence of the BLV on the number of the plates in the shield assuming that all plates have the same thicknesses. The latter assumption implies the following expression for the BLV of the layered shield:

$$v_{bl}^2 = g(N) = NG(T_{sum}/N). \tag{13}$$

Assuming that N varies continuously we can calculate the derivative:

$$g'(N) = G(T) - TG'(T), \quad T = T_{sum}/N. \tag{14}$$

It is proved in Appendix A that Eq. (8) implies Eq. (A.1) when $G(T)$ is a convex function. Equation (A.1), in its turn, implies that $g'(N) < 0$ if $T_{sum}/N > T_{\min}$. Therefore $g(N)$ decreases when

$$1 < N \leqslant T_{sum}/T_{\min}. \tag{15}$$

The conclusion is that the BLV decreases with the increase of the number of layers if all plates have the same thicknesses and $G(T)$ is a convex function.

In the case of a linear function $G(T)$, Eq. (12) implies that the BLV decreases with the increase of the number of layers ($N < -\mu_1 T_{sum}/\mu_0$), independent of their thicknesses if $\mu_0 < 0$ and does not depend on of the number of layers if $\mu_0 = 0$.

6. Discussion of assumptions and results

There are only a few publications which can be used for the analysis of behavior of the function $G(T)$.

The model of Ohte et al. [21] for conical-nosed impactors implies that

$$v_{bl}^2 = k_1 T^3, \quad k_1 = \frac{\beta}{m}\alpha^{1.5}, \quad \alpha = 1 + 2.9\left(\tan\frac{\Theta}{2}\right)^{2.1}, \quad \frac{T}{D} < \frac{1}{\alpha}, \tag{16}$$

where m and D are the mass and the diameter of the impactor, correspondingly, Θ is the nose angle for the conical-nosed projectile, T is the thickness of the shield, and the coefficient β depends on the system of units. Therefore, $G(T) = k_1 T^3$ is a convex function.

Gupta et al. [22] describes the dependence between the BLV of the ogive-nosed impactor and the thickness of the aluminum plate as $v_{bl} = k_2 T^{0.89}$, where k_2 is a constant. In this case, $G(T) = k_2 T^{1.78}$ is a convex function. Functions $G(T)$ in Fig. 1a, which we plotted using experimental results [6,14], are convex and satisfy the condition given by Eq. (8). Processing of the experimental results [23] (Fig. 1b) yields a linear function $G(T)$, with $\mu_0 < 0$ in three out of four cases and $\mu_0 = 0$ in one case.

Results of the experiment and numerical calculations [5–8,13–15] used in the analysis of our theoretical predictions are summarized in Tables 1–8. All experiments/calculations in each table are obtained for the impactor having the same shape against the shield manufactured from the same material.

The structure of the shield is shown as follows. In the case of a monolithic shield, the thickness of the shield is indicated. The notation 2 × 6.0 denotes the shield which consists of two plates having the same thicknesses, 6.0 mm. The notation 6.0 + 2.0 denotes the two-layer shield having the front plate with the thickness 6.0 mm and the rear plate with the thickness 2.0 mm. In each column, the data concerning the monolithic shield is given in the first row, while data on the layered shields are shown below.

Tables 1–3 contain data on pairs, impact velocity–residual velocity. Of two shields, the shield with a smaller residual velocity and the same impact velocity is assumed as having superior protective characteristics. Tables 4–8 contain data on the BLVs whereby the superior shield has a larger value of the BLV.

Comparison of the theoretical results obtained in Section 3 with experimental and numerical results is shown in Tables 9 and 10. Inequality $\eta_{res} < 0$ or $\eta_{bl} < 0$, where

$$\eta_{res} = \frac{v_{res} - V_{res}}{v_{res}}, \quad \eta_{bl} = \frac{V_{bl} - v_{bl}}{v_{bl}}, \tag{17}$$

indicates the agreement between theoretical predictions and experimental or numerical results and $\eta_{res} > 0$ or $\eta_{bl} > 0$ denotes the discrepancy from the theoretical predictions. Inspection of Tables 9 and 10 reveals that theoretical predictions are supported by experimental or numerical data in 87% of cases.

Inspection of Table 3 shows that for all values of the impact velocity, the residual velocity for the two-layered shield with the same thicknesses of the layers (2 × 4.0 mm) is larger than for any other partitioning of the shield, in compliance with the theoretical predictions in Section 4 for the case of convex function $G(T)$. However, the BLV in Table 8 for 2 × 4.76 mm shield is larger than for other two variants of the shield with unequal thicknesses of the layers. The effect of the thicknesses of the layers upon ballistic properties of the shield requires further investigations. Indeed, the maximum adverse effect of layering with equal thicknesses of the layers in our analysis is the consequence of the convexity of the function $G(T)$, while the available experimental data afford also the linear function $G(T)$, whereby the ballistic efficiency of the shield does not depend upon the thicknesses of the layers.

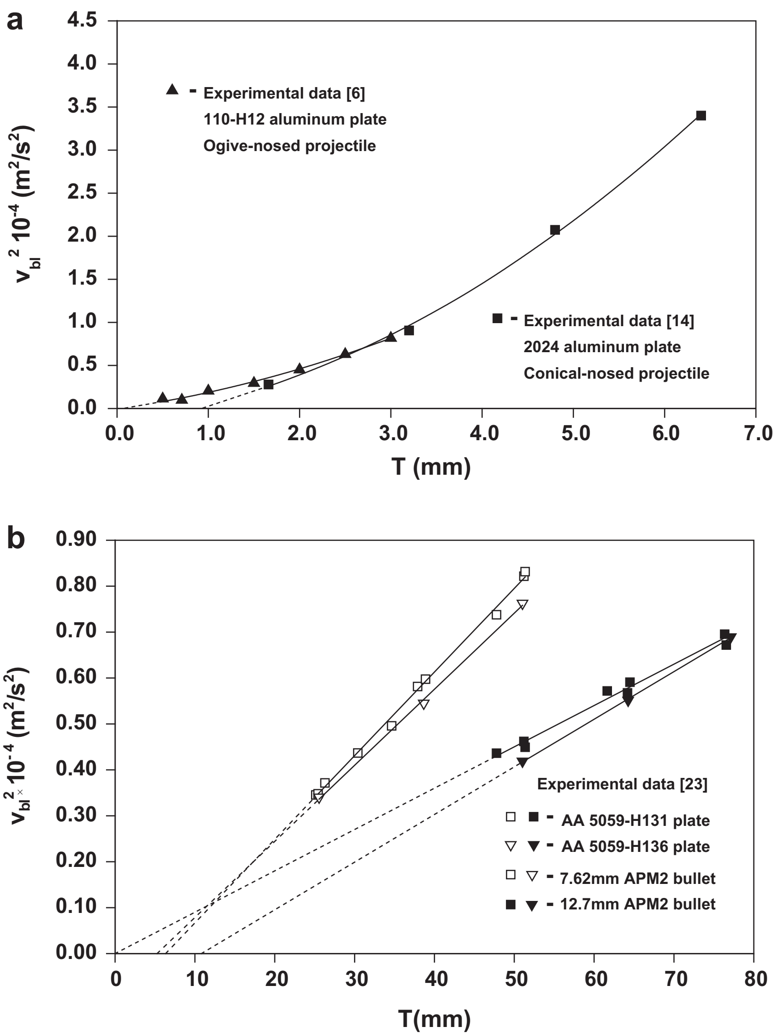

Fig. 1. (a and b) Squared BLV vs. thickness of the shield. Dotted lines are either tangents to the curves at the point $T = T_{\min}$ or continuations of the straight lines to the left of this point.

Table 1
Numerical results of Iqbal et al. [5] on residual velocities (conical-nosed projectile, Weldox 460E steel shield).

No.	Shield (sizes in mm)	Impact velocity (m/s)					
		600	406	356	318	300	281
1	12.0	555	333	269	217	189	154
2	2 × 6.0	554	337	273	221	195	163

Table 2
Numerical results of Iqbal et al. [5] on residual velocities (ogive-nosed projectile, 1100–H12 aluminum shield).

No.	Shield (sizes in mm)	Impact velocity (m/s)							
		150	113	97	83	82	73	66	57
1	1.0	141	100	81	64	62	49	32	15
2	2 × 0.5	145	101	84	67	65	54	40	20

Comparison of different structures of the shield 7-8 in Table 4, 6-7 and 11-12 in Table 5 and 4-5-6 in Table 4 supports the theoretical predictions in Section 5: increase of the number of layers having the same thickness impairs ballistic properties of the shield.

Table 3
Experimental results of Almohandes et al. [13] on residual velocities (7.62 mm bullet, mild steel shield).

No.	Shield (sizes in mm)	Impact velocity (m/s)				
		706	755	775	805	826
1	8.0	446	489	528	611	647
2	2.0 + 6.0	477	514	549	609	658
3	6.0 + 2.0	471	522	–	616	654
4	2 × 4.0	486	533	567	643	667

Table 4
Experimental results of Radin and Goldsmith [14] on BLV (conical-nosed projectile, 2024- Aluminum shield).

No.	Shield (sizes in mm)	BLV (m/s)	No.	Shield (sizes in mm)	BLV (m/s)	No.	Shield (sizes in mm)	BLV (m/s)
1	3.2	95	3	4.8	144	6	6.4	184
2	2 × 1.6	93	4	1.6 + 3.2	130	7	2 × 3.2	160
			5	3 × 1.6	124	8	4 × 1.6	158

Table 5
Experimental results of Iqbal et al. [6] on BLV (ogive-nosed projectile, 1100–H12 aluminum shield).

No.	Shield (sizes in mm)	BLV (m/s)	No.	Shield (sizes in mm)	BLV (m/s)	No.	Shield (sizes in mm)	BLV (m/s)	No.	Shield (sizes in mm)	BLV (m/s)	No.	Shield (sizes in mm)	BLV (m/s)
1	1.0	45.3	3	1.5	54.3	5	2.0	67.1	8	2.5	79.3	10	3.0	90.4
2	2 × 0.5	39.2	4	3 × 0.5	51.0	6	2 × 1.0	62.9	9	5 × 0.5	66.0	11	2 × 1.5	78.6
						7	4 × 0.5	61.7				12	3 × 1.0	74.2

Table 6
Numerical results of Teng et al. [7] on the BLV (conical-nosed projectile, Weldox 460E steel shield).

Conical-nosed projectile			Light conical-nosed projectile		
No.	Shield (sizes in mm)	BLV (m/s)	No.	Shield (sizes in mm)	BLV (m/s)
1	12.0	305.9	3	12.0	525.9
2	2 × 6.0	282.0	4	2 × 6.0	524.5

Table 7
Experimental results of Dey et al. [8] on the BLV (ogive-nosed projectile, Weldox 460E steel shield).

No.	Shield (sizes in mm)	BLV (m/s)
1	12.0	318.1
2	2 × 6.0	288.3

Table 8
Experimental results of Woodward et al. [15] (conical-nosed projectile, 2024-T351 aluminum alloy shield).

No.	Shield (sizes in mm)	BLV (m/s)
1	9.53	462
2	3.18 + 6.35	421
3	6.35 + 3.18	474
4	2 × 4.76	484
5	3 × 3.18	473
6	6 × 1.59	454

Therefore, the available experimental and numerical results support our theoretical predictions.

7. Concluding remarks

On the basis of general assumptions that are pertinent to penetration modeling, we investigated the effect of layering of metallic shields against sharp-nosed rigid impactors and established/explained a number of general assertions concerning

Table 9
Validation of theoretical results on the residual velocity.

Table	Compared shields	v_{imp} (m/s)	$\frac{v_{res}-V_{res}}{v_{res}}$
1	1-2	600	+0.002
1	1-2	406	−0.012
1	1-2	356	−0.015
1	1-2	318	−0.018
1	1-2	300	−0.032
1	1-2	281	−0.059
2	1-2	150	−0.028
2	1-2	113	−0.010
2	1-2	97	−0.037
2	1-2	83	−0.047
2	1-2	82	−0.048
2	1-2	73	−0.068
2	1-2	66	−0.250
2	1-2	57	−0.333
3	1-2	706	−0.070
3	1-3	706	−0.056
3	1-4	706	−0.090
3	1-2	755	−0.051
3	1-3	755	−0.067
3	1-4	755	−0.090
3	1-2	775	−0.040
3	1-4	775	−0.074
3	1-2	805	+0.003
3	1-3	805	−0.008
3	1-4	805	−0.052
3	1-2	826	−0.017
3	1-3	826	−0.011
3	1-4	826	−0.031

Table 10
Validation of theoretical results on the BLV.

Table	Compared shields	$\frac{V_{bl}-v_{bl}}{v_{bl}}$
4	1-2	−0.021
4	3-4	−0.097
4	3-5	−0.139
4	6-7	−0.130
4	6-8	−0.165
5	1-2	−0.135
5	3-4	−0.061
5	5-6	−0.063
5	5-7	−0.080
5	8-9	−0.168
5	10-11	−0.131
5	10-12	−0.204
6	1-2	+0.078
6	3-4	+0.003
7	1-2	+0.094
8	1-2	+0.089
8	1-3	−0.026
8	1-4	−0.048
8	1-5	−0.024
8	1-6	+0.017

ballistic properties of the shields. Omitting the technical details, the main conclusions of this study can be formulated as follows:

(1) Layering impairs the protective properties of the shield against perforation.
(2) Increasing the number of layers having the same thicknesses impairs protective properties of the shield.

Although our theoretical predictions are confirmed by comparing with experimental or numerical results it will be useful to conduct a dedicated experimental study in order to determine the range of validity of our theoretical prediction depending on the material properties of the shield, shape of the impactor, thicknesses of the layers and impact velocity.

Appendix A. Proof of Claim

In order to prove the claim we use the method of mathematical induction.

As the first step, we prove that the claim is valid for the case of two plates ($N = 2$).

Consider the function $F(T) = TG'(T) - G(T)$. Since $G(T)$ is a convex function, $F'(T) = TG''(T) > 0$ and, consequently,

$$TG'(T) - G(T) > 0, \quad T > T_{\min}. \tag{A.1}$$

Assume that $T_1 \geqslant T_2$ (if $T_1 < T_2$ we need only to replace $T_1 \leftrightarrow T_2$). Then using the mean value theorem and Eq. (A.1) we obtain:

$$\begin{aligned}\delta &= G(T_1 + T_2) - G(T_1) - G(T_2) = [G(T_1 + T_2) - G(T_1)] - G(T_2) = T_2 G'(\xi) - G(T_2) > T_2 G'(\xi) - T_2 G'(T_2) \\ &= T_2[G'(\xi) - G'(T_2)] = T_2 G''(\theta),\end{aligned} \tag{A.2}$$

where

$$T_1 < \xi < T_1 + T_2, \quad T_2 < \theta < \xi. \tag{A.3}$$

Since $\theta > T_{\min}$, we proved that $\delta > 0$.

At the second step, we assume that the claim is valid for N plates ($\delta = \delta_N > 0$) and prove that it is valid for $N + 1$ plates ($\delta_{N+1} > 0$). Using the validity of the statement for two plates with the thicknesses $T^{(1)} + T^{(2)} + \cdots + T^{(N)}$ and $T^{(N+1)}$ we can write:

$$G\left(\sum_{j=1}^{N+1} T^{(j)}\right) = G\left(\sum_{j=1}^{N} T^{(j)} + T^{(N+1)}\right) > G\left(\sum_{j=1}^{N} T^{(j)}\right) + G\left(T^{(N+1)}\right). \tag{A.4}$$

Then using the validity of the claim for N plates we obtain:

$$\delta_{N+1} = G\left(\sum_{j=1}^{N+1} T^{(j)}\right) - \sum_{j=1}^{N+1} G(T^{(j)}) > G\left(\sum_{j=1}^{N} T^{(j)}\right) + G(T^{(N+1)}) - \sum_{j=1}^{N+1} G(T^{(j)}) = G\left(\sum_{j=1}^{N} T^{(j)}\right) - \sum_{j=1}^{N} G(T^{(j)}) > 0. \tag{A.5}$$

Appendix B. The worst layering

In this appendix we prove that the worst layering occurs when the monolithic shield is replaced by the identical plates having the same total thickness.

After eliminating the variable $T^{(N)}$, the problem can be formulated as follows:

$$\Lambda(T^{(1)}, \ldots, T^{(N-1)}) = \sum_{j=1}^{N-1} G(T^{(j)}) + G(T_{sum} - \sum_{j=1}^{N-1} T^{(j)}) \to \min \tag{B.1}$$

subjected to the following constraints:

$$\sum_{j=1}^{N-1} T^{(j)} \leqslant T_{sum} - T_{\min}, \quad T^{(i)} \geqslant T_{\min}, \ i = 1, \ldots, N-1. \tag{B.2}$$

Let us prove that Λ is a convex function. Formulas for first and the second order derivatives of G ($\nu, \mu = 1, \ldots, N-1$) read:

$$H^{(\nu)} \equiv \frac{\partial \Lambda}{\partial T^{(\nu)}} = G'(T^{(\nu)}) - G'(T^{(N)}), \quad T^{(N)} = T_{sum} - \sum_{j=1}^{N-1} T^{(j)}, \tag{B.3}$$

$$H^{(\nu,\mu)} \equiv \frac{\partial^2 \Lambda}{\partial T^{(\nu)} \partial T^{(\mu)}} = \begin{cases} G''(T^{(N)}) & \text{if} \quad \nu \neq \mu \\ G''(T^{(\nu)}) + G''(T^{(N)}) & \text{if} \quad \nu = \mu \end{cases}. \tag{B.4}$$

Then the characteristic equation,

$$\begin{vmatrix} H^{(1,1)} - \eta & H^{(1,2)} & \ldots & H^{(1,N-1)} \\ H^{(2,1)} & H^{(2,2)} - \eta & \ldots & H^{(2,N-1)} \\ \ldots & \ldots & \ldots & \ldots \\ H^{(N-1,1)} & H^{(N-1,2)} & \ldots & H^{(N-1,N-1)} - \eta \end{vmatrix} = 0, \tag{B.5}$$

can be written as follows:

$$\prod_{j=1}^{N-1} [G''(T^{(\nu)}) - \eta] = 0 \tag{B.6}$$

because the last term in Eq. (B.4) is the same for all $H^{(\nu,\mu)}$ and can be dropped out in the determinant in Eq. (B.5), and the nonzero elements remain only at the diagonal. Since $G''(T) > 0$, all the roots of Eq. (B.6) are positive and, consequently, Λ is a convex function. Taking into account that the domain determined by the linear inequalities in Eq. (B.2) is convex, we conclude that the function Λ has the unique minimum. The point of minimum can be determined using the conditions $H^{(\nu)} = 0(\nu = 1, \ldots, N-1)$. By virtue of Eq. (B.3) these conditions imply that

$$G'(T^{(1)}) = G'(T^{(2)}) = \cdots = G'(T^{(N-1)}) = G'(T^{(N)}). \tag{B.7}$$

The solution of Eqs. (B.7) is $T^{(1)} = T^{(2)} = \cdots = T^{(N-1)} = T^{(N)} = T_{sum}/N$.

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